GROUP PROPERTIES OF MHD EQUATIONS AND THEIR INVARIANT SOLUTIONS

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We perform a group analysis of the equations of magnetohydrodynamics. As distinct from [1, 2], the group properties of the equations of motion of a compressible fluid are considered under the assumption of finite conductivity. Possible invariant solutions are found for the set of MHD equations in the one-dimensional case. We give examples of analytic and numerical solutions of the problem of conducting gas flow interaction with a magnetic field.

The set of equations describing the nonstationary flow of an electrically conducting gas in a magnetic field in the hydrodynamic approximation (displacement currents are neglected throughout) is

$$\begin{gathered}
\frac{\partial \mathbf{h}}{\partial t} = \operatorname{rot}[\mathbf{v}\times\mathbf{h}] - \operatorname{rot}(\nu_m \operatorname{rot}\mathbf{h}),\\
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\operatorname{grad} p + \frac{1}{4\pi\rho}(\operatorname{rot}\mathbf{h}\cdot\mathbf{h}) + \frac{1}{\rho}\operatorname{div}\boldsymbol{\tau},\\
\frac{\partial p}{\partial t} + (\mathbf{v}\cdot\operatorname{grad} p) + \gamma p \operatorname{div}\mathbf{v} = \frac{\gamma-1}{4\pi}\nu_m(\operatorname{rot}\mathbf{h})^2 - (\gamma-1)\operatorname{div}\mathbf{q} + (\gamma-1)F,\\
\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{v}) = 0, \qquad p = R\rho T, \qquad \operatorname{div}\boldsymbol{\tau} = \frac{\partial \tau_{ik}}{\partial x_k}\mathbf{e}_i,\\
\tau_{ik} = \mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\frac{\partial v_j}{\partial x_j}\delta_{ik}\right) + \xi\frac{\partial v_j}{\partial x_j}\delta_{ik},\\
\mathbf{q} = -\lambda \operatorname{grad} T, \qquad F = \frac{\tau_{ik}}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right), \qquad \nu_m = \frac{c^2}{4\pi\sigma}.
\end{gathered} \tag{S_1}$$

The conductivity σ, thermal conductivity λ, and gas viscosity coefficients μ and ξ are functions of p and the density ρ:

$$\sigma = ap^m\rho^n, \qquad \lambda = bp^\omega\rho^\psi, \qquad \mu = dp^g\rho^\varphi, \qquad \xi = fp^g\rho^\varphi. \tag{1}$$

Let us consider the group properties of the system S_1 of differential equations under condition (1) in three-dimensional space in which the velocity vector v has components v_1, v_2, and v_3 and the magnetic field strength vector h has components h_1, h_2, and h_3. As is well known, the transformation group G of the system of differential equations is completely defined by the Lie algebra of its infinitesimal operators. Using familiar methods [3] we find that the Lie algebra of the fundamental group of system S_1 under condition (1) is generated by the following linearly independent operators:

$$\begin{gathered}
X_1 = \frac{\partial}{\partial t}, \qquad X_2^i = \frac{\partial}{\partial x_i}, \qquad X_3^i = t\frac{\partial}{\partial x_i} + \frac{\partial}{\partial v_i} \qquad (i, k = 1, 2, 3),\\
X_{ik} = x_i\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_i} + v_i\frac{\partial}{\partial v_k} - v_k\frac{\partial}{\partial v_i} \qquad (i < k).
\end{gathered} \tag{2}$$

In the case in which only molecular heat conduction is present ($\psi = \varphi$, $\omega = g$), further extension of the group occurs when $n \neq -m$; to the operators (2) we add

$$\begin{gathered}
X_4 = [\alpha(1-n)+1]\,t\frac{\partial}{\partial t} + \alpha\frac{1-2n}{2}x_i\frac{\partial}{\partial x_i} - \left(\frac{\alpha}{2}+1\right)v_i\frac{\partial}{\partial v_i} +\\
+ \frac{1-2n}{n+m}p\frac{\partial}{\partial p} + \left(\alpha + \frac{2m+1}{n+m}\right)\rho\frac{\partial}{\partial\rho} + \frac{1-2n}{2(n+m)}h_i\frac{\partial}{\partial h_i},\\
\alpha = \frac{(m-n+1) - \psi(2m+1) + \omega(2n-1)}{(n+m)(n+\psi-1)},\\
X_5 = [\alpha(1-n)-m+1]\,t\frac{\partial}{\partial t} + \left(\frac{1-2n}{2}\alpha + \frac{2m+1}{2}\right)x_i\frac{\partial}{\partial x_i} +\\
+ \frac{\alpha-1}{2}v_i\frac{\partial}{\partial v_i} + p\frac{\partial}{\partial p} + \alpha\rho\frac{\partial}{\partial\rho} - \frac{\alpha(1-2n)-1}{2}h_i\frac{\partial}{\partial h_i},\\
\alpha = \frac{m+\omega}{1-\psi-n}.
\end{gathered} \tag{3}$$

When 2m = −n, we add to these last two operators

$$X_6=[\alpha(1-n)+2]\,t\frac{\partial}{\partial t}+\left(\frac{1-2n}{2}\alpha+1\right)x_i\frac{\partial}{\partial x_i}-\left(\frac{\alpha}{2}+1\right)v_i\frac{\partial}{\partial v_i}-$$
$$-4p\frac{\partial}{\partial p}+(\alpha-2)\rho\frac{\partial}{\partial\rho}-2h_i\frac{\partial}{\partial h_i},\qquad \alpha=\frac{2\omega+\psi-1}{2(1-n-\psi)}.\tag{4}$$

When n = −m, the operators X_4, X_5, and X_6 do not apply, and we add to (2) the operator

$$X_7=\frac{2m+2}{2m+1}t\frac{\partial}{\partial t}+x_i\frac{\partial}{\partial x_i}+\left(\alpha-\frac{2}{2m+1}\right)p\frac{\partial}{\partial p}+\alpha\rho\frac{\partial}{\partial\rho}-$$
$$-\frac{1}{2m+1}v_i\frac{\partial}{\partial v_i}+\left(\frac{\alpha}{2}-\frac{1}{2m+1}\right)h_i\frac{\partial}{\partial h_i},\qquad \alpha=\frac{2\omega+2m}{(2m+1)(\omega+\psi-1)}.\tag{5}$$

If we are considering radiative heat transfer, where the conditions $\psi=\varphi$, $\omega=g$ do not hold, the operators X_4–X_6 are only applicable for the system of equations S_1, in which the viscosity terms are discounted. If we discount the heat conduction terms in system S_1 but retain the viscosity terms, the operators X_4–X_6 again hold, except that we have to replace ω and ψ by g and φ, respectively, in the expressions for α.

If we consider the movement of a nonviscous electrically conducting gas and disregard the heat conduction, further extension of the Lie algebra of the fundamental group of system S_1 occurs provided that $\sigma=ap^m\rho^n$. To the operators (2)–(5), in which we set α equal to zero, we add:

when $n\neq -m$,

$$X_8=(1+m)\,t\frac{\partial}{\partial t}+\frac{1+2m}{2}x_i\frac{\partial}{\partial x_i}-\frac{1}{2}v_i\frac{\partial}{\partial v_i}+\rho\frac{\partial}{\partial\rho};\tag{6}$$

when $\gamma=2$ and 2m = −n,

$$X_9=t^2\frac{\partial}{\partial t}+x_it\frac{\partial}{\partial x_i}+(v_it-x_i)\frac{\partial}{\partial v_i}+4tp\frac{\partial}{\partial p}+2t\rho\frac{\partial}{\partial\rho}+2th_i\frac{\partial}{\partial h_i}.\tag{7}$$

Extension of the group also occurs when n = −m:

$$X_{10}=p\frac{\partial}{\partial p}+\rho\frac{\partial}{\partial\rho}+\frac{1}{2}h_i\frac{\partial}{\partial h_i}\tag{8}$$

Let us consider in more detail the case of uniform movement of a nonviscous electrically conducting gas in a magnetic field. We neglect heat conduction, and denote by S_2 the system of equations describing this flow. We assume that n = −m. This means that, under the above assumptions, the gas conductivity is given as a function of temperature by $\sigma=aT^m$. Under our assumptions the system S_2 is invariant with respect to the operators

$$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x},\quad X_3=t\frac{\partial}{\partial x}-\frac{\partial}{\partial v},\quad X_4=p\frac{\partial}{\partial p}+\rho\frac{\partial}{\partial\rho}+\frac{1}{2}h\frac{\partial}{\partial h},$$
$$X_5=\frac{2m+2}{2m+1}t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-\frac{1}{2m+1}v\frac{\partial}{\partial v}-\frac{2}{2m+1}p\frac{\partial}{\partial p}-\frac{1}{2m+1}h\frac{\partial}{\partial h}.\tag{9}$$

These operators apply in the case of plane flows. For axisymmetric flows a contraction of the group occurs and only the three linearly independent operators X_1, X_4, and X_5 remain. A knowledge of the fundamental group (9) enables the invariant solutions of system S_2 to be found. Invariant solutions of unit rank are only possible for S_2 in single-parameter subgroups. Utilizing the internal automorphisms of the transformation group G, we can obtain an optimum system of single-parameter subgroups, whence all the essentially distinct solutions of system S_2 can be found.

We omit the intermediate steps and present only the final expressions for the optimum system of single-parameter subgroups:

$$X_1+\beta X_4,\ X_5+\beta X_4,\ X_2+\beta X_4,\ X_3+\beta X_4,\ X_1+X_3+\beta X_4,\tag{10}$$

where β is an arbitrary constant.

Using the optimum subgroups (10), we obtain the corresponding essentially distinct invariant solutions.

1. Subgroup H_1 with operator $X_1 + \beta X_4$. The invariant H_1-solution is

$$v = V(x),\ p = e^{2\beta t}P(x),\ \rho = e^{2\beta t}\theta(x),\ h = e^{\beta t}\Phi(x).$$

2. The subgroup H_2 with operator $X_5 + \beta X_4$. The invariant H_2-solution can be written as

$$v = \frac{x}{t}V(\lambda),\quad p = x^{\beta - 2/(2m+1)}P(\lambda),\quad \rho = x^{\beta}\theta(\lambda),$$
$$h = x^{\beta/2 - 1/(2m+1)}\Phi(\lambda),\quad \lambda = tx^{-(2m+2)/(2m+1)}.$$

3. The subgroup H_3 with operator $X_2 + \beta X_4$. The invariant H_3-solution is

$$v = V(t),\ p = e^{2\beta x}P(t),\ \rho = e^{2\beta x}\theta(t),\ h = e^{\beta x}\Phi(t).$$

4. The subgroup H_4 with operator $X_3 + \beta X_4$. The invariant H_4-solution is

$$v = x/t + V(t),\ p = e^{2\beta x/t}P(t),\ \rho = e^{2\beta x/t}\theta(t),\ h = e^{\beta x/t}\Phi(t).$$

5. The subgroup H_5 with operator $X_1 + X_3 + \beta X_4$. The invariant H_5-solution may be written as

$$v = t + V(\lambda),\ p = e^{2\beta t}P(\lambda),\ \rho = e^{2\beta t}\theta(\lambda),\ h = e^{\beta t}\Phi(\lambda),\ \lambda = x - 1/2t^2.$$

The functions V, P, θ, and Φ satisfy, respectively, the systems of ordinary differential equations obtained by direct substitution of the expressions for v, p, ρ, and h into S_2. Numerical methods may be used for finding the solutions of these systems of equations. However, if the magnetic pressure is proportional to the static gas pressure, an analytic solution of the problem may easily be found in the subgroups H_3 and H_4.

The invariant H_3-solution, with m = 3/2, is

$$\begin{gathered} v = -\frac{N}{M}(Mt + C_1)^{5/3} + C_2,\ \rho = C_3\exp\left[\frac{3}{8}\beta\frac{N}{M^2}(Mt + C_1)^{8/3} - \beta C_2 t + \beta x\right], \\ p = \frac{\gamma - 1}{8\pi}C_3(Mt + C_1)^{2/3}\exp\left[\frac{3}{8}\beta\frac{N}{M^2}(Mt + C_1)^{8/3} - \beta C_2 t + \beta x\right], \\ h = C_3^{1/2}(Mt + C_1)^{1/3}\exp\left[\frac{3}{16}\beta\frac{N}{M^2}(Mt + C_1)^{8/3} - \frac{\beta}{2}C_2 t + \frac{\beta}{2}x\right], \\ M = \frac{3}{4}\beta^2 a\left(\frac{8\pi}{\gamma - 1}\right)^{3/2},\qquad N = \frac{3}{5}\frac{\gamma\beta}{8\pi}. \end{gathered} \tag{11}$$

The invariant H_4-solution, with $\gamma = 2$ and m = 1, is

$$\begin{gathered} v = \frac{x}{t} - N\left[\frac{M}{2}\frac{(\ln t)^2}{t} + C_1\frac{\ln t}{t}\right] + C_2, \\ \rho = C_3 t^{2f(t)}\exp\left[\beta\frac{x}{t} - \frac{\beta N(M + C_1)}{t}\right], \\ p = C_3\frac{1}{8\pi}\left(M\frac{\ln t}{t} + \frac{C_1}{t}\right)t^{2f(t)}\exp\left[\beta\frac{x}{t} - \frac{\beta N(M + C_1)}{t}\right], \\ h = C_3^{1/2}\left(M\frac{\ln t}{t} + \frac{C_1}{t}\right)t^{f(t)}\exp\left[\frac{\beta}{2}\frac{x}{t} - \frac{\beta N(M + C_1)}{2t}\right], \\ M = 4\pi a\beta^2,\quad N = \frac{\beta}{4\pi},\quad f(t) = \frac{1 + \beta C_2}{2} - \frac{\beta MN}{4}\frac{\ln t}{t} - \frac{\beta N(M + C_1)}{2t}. \end{gathered} \tag{12}$$

The constants C_1, C_2, and C_3 are found from the initial conditions.

In conclusion, let us use our invariant H_1-solution to consider the radial flow of a gas of finite conductivity in a longitudinal magnetic field. We take a combination of an infinite cylindrical source of electrically conducting gas of radius R_1 and a sink of radius $R_2 > R_1$, and consider the gas movement in the magnetic field of an infinite solenoid of radius R_2. In view of the form of the H_1-solution, we get the time dependence $I = I_0 e^{\beta t}$ for the current in the solenoid. In addition, we assume that, when $r \le R_1$, the conductivity σ tends to infinity, i. e., the electric field strength vanishes.

Substituting the expressions for v, p, ρ, and h from the H_1-solution into S_2, we get a system of equations in the functions V(r), P(r), θ(r), and Φ(r):

$$\beta\Phi + \frac{1}{r}\frac{\partial}{\partial r}(rv\Phi) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\nu_m \frac{\partial\Phi}{\partial r}\right)$$

$$V\frac{\partial V}{\partial r} = -\frac{1}{\theta}\frac{\partial}{\partial r}\left(p + \frac{\Phi^2}{8\pi}\right), \quad 2\beta\theta + \frac{1}{r}\frac{\partial}{\partial r}(rV\theta) = 0 \tag{13}$$

$$2\beta P + V\frac{\partial P}{\partial r} + \gamma P\left(\frac{\partial V}{\partial r} + \frac{V}{r}\right) = (\gamma - 1)\frac{\nu_m}{4\pi}\left(\frac{\partial\Phi}{\partial r}\right)^2$$

The boundary conditions are

$$V\,|_{r=R_1} = V_0, \qquad P\,|_{r=R_1} = P_0, \qquad \theta\,|_{r=R_1} = \theta_0,$$
$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=R_1} = \frac{4\pi\sigma}{c^2} V\Phi\,|_{r=R_1}, \qquad \Phi\,|_{r=R_2} = \frac{4\pi\delta}{c} I_0 . \tag{14}$$

Here δ is the number of turns per unit length of the solenoid. The fourth condition is obtained from Ohm's law. We have thus obtained a boundary value problem for the system (13) under conditions (14).

To maintain the specified current $I = I_0 e^{\beta t}$ in the solenoid, a suitable emf E must be included in the electrical circuit containing the solenoid. This emf is found from the equation

$$E = I_0 \Omega e^{\beta t} + 2\pi\beta\delta e^{\beta t} \int_{R_1}^{R_2} r\Phi(r)\,dr ,$$

of the electrical circuit, where Ω is the circuit resistance.

The problem was solved on a computer, taking m = 3/2. The results confirm the formation of a high-temperature electrically conducting layer, as indicated in [4, 5]. The formation of this high-temperature layer is accompanied by a sharp braking of the gas in this zone: see Fig. 1, where the following notation is used for the dimensionless quantities:

$$v_1 = \frac{v}{v_0}, \qquad T_1 = \frac{T}{T_0}, \qquad A = \frac{2\pi\delta^2 I_0^2}{c^2 p_0}, \qquad r_1 = \frac{r}{R_1}$$

Here, ν_0, T_0, p_0, and I_0 are the characteristic values of the velocity, temperature, pressure, and current.

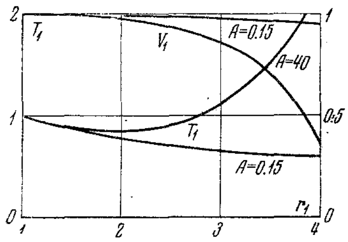

Fig. 1

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